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A fluctuation–relaxation relation for homogeneous, isotropic turbulence

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Abstract

A Liouville-type equation is solved perturbatively for the exact pdf of fluctuating velocities by expansion about an invariant Gaussian measure, which is an approximation to the *symmetric part* of the pdf and is constrained to give the correct covariance. The *antisymmetric* part of the pdf is calculated to the lowest nontrivial order and leads to closed equations for the covariance in terms of a response function which arises from the Liouville operator averaged against the ground-state distribution. As the latter is a stationary Gaussian pdf, we obtain an expression for the system response which is linear in the covariance.

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1. Introduction

The bedrock problem in the statistical theory of turbulence, as in all stochastic many-body problems, is the fact that the moments form an open hierarchy. This *closure* problem received some attention in the early 1960s, with the development of renormalized perturbation theories [1–5], which were physically realizable but unfortunately were found not to be compatible with the Kolmogorov $-5/3$ spectral law at large Reynolds numbers (K41).

Various explanations were given for this failure, but it was later shown [6] that *all* these theories have an incorrect interpretation of energy conservation in terms of their response functions. This arises due to an incorrect division of the energy transfer spectrum into input and output parts, rather as if in a Fermi master equation. In contrast, the local energy transfer (LET) theory [6] used a local (in wavenumber) energy balance to determine the response function, in which the entire transfer spectrum acted as an input or an output, according to the value of the wavenumber. This is in accord with experiment and was later extended to derive a two-time form of LET theory [7]. Both these forms of the theory are compatible with K41.

We can sum up the existing state of turbulence theory by pointing to the existence of the Kraichnan–Wyld covariance equations. These equations represent an exact second-order truncation of renormalized perturbation theory. Originally derived heuristically by Kraichnan

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[1], their derivation can rely either on the topology of Feynmann-type diagrams [2] or on reversion of power series [8]. They satisfy all the required symmetries and, in particular, they conserve energy, displaying the correct antisymmetric behaviour of the transfer spectrum. With an appropriate choice of renormalized response function R they can predict the free decay of isotropic turbulence, without invoking arbitrary constants, in broadly good agreement with experiment and numerical simulation.

The turbulence problem in this context becomes one of finding a principle to determine the renormalized response function R, and it was suggested [9, 10] that the classical fluctuation–dissipation relation (FDR) could be used for this purpose. Introducing the covariance C and response function R in wavenumber, this takes the well-known form

$$R(k; t, t')C(k; t', t') = C(k; t, t'). \tag{1}$$

Also, it was later realized that the LET theory could be derived retrospectively from the Kraichnan–Wyld equations by introducing the above relationship: see [11] for a discussion and references therein, and also [12] for an interpretation of closures in terms of the FDR.

While there is a body of evidence that (1) can be used in a pragmatic way to study macroscopic, non-Hamiltonian problems, there is also some unease with the fact that the *derivation* of this simple linear relationship is not itself applicable to such systems. It was therefore a welcome development, when it was shown that a general fluctuation–response relationship could be obtained for chaotic dynamical systems which were mixing, in which the response was related to the invariant measure of the system [13, 14]. It was also shown that this general fluctuation–response relationship reduced to (1) for the case of Gaussian invariant measure. For a further discussion of these topics, see [15, 16]: at this point we shall merely remark that fluid turbulence is inherently non-Gaussian, so that it is by no means obvious that (1) can apply.

In this paper we follow in the footsteps of Edwards [3] and solve the Liouville equation perturbatively for the turbulence probability distribution functional. We extend the analysis from the single-time stationary form [3] to the non-stationary two-time form. In the process, we show that (1) holds to the lowest nontrivial order in perturbation theory.

2. The basic equations

Here we introduce the general notation and the basic equations. A general discussion of these equations may be found in [17]. Consider the solenoidal Navier–Stokes equation (NSE) in wavenumber (k) space:

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) u_\alpha(\mathbf{k}, t) = M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3 j\, u_\beta(\mathbf{j}, t) u_\gamma(\mathbf{l}, t) + f_\alpha(\mathbf{k}, t), \tag{2}$$

where ν is the fluid kinematic viscosity, $\mathbf{l} = \mathbf{k} - \mathbf{j}$, $f_\alpha(\mathbf{k}, t)$ is an arbitrarily chosen stirring force which we have to specify, and the inertial transfer operator $M_{\alpha\beta\gamma}(\mathbf{k})$ is given by $M_{\alpha\beta\gamma}(\mathbf{k}) = (2i)^{-1}[k_\beta P_{\alpha\gamma}(\mathbf{k}) + k_\gamma P_{\alpha\beta}(\mathbf{k})]$, while the projector $P_{\alpha\beta}(\mathbf{k})$ is expressed in terms of the Kronecker delta as $P_{\alpha\beta}(\mathbf{k}) = \delta_{\alpha\beta} - k_\alpha k_\beta/k^2$. The covariance of the fluctuating velocity field may be introduced as

$$C_{\alpha\beta}(\mathbf{k}, \mathbf{k}'; t, t') \equiv \langle u_\alpha(\mathbf{k}, t) u_\beta(\mathbf{k}', t')\rangle = P_{\alpha\beta}(\mathbf{k})\delta(\mathbf{k} + \mathbf{k}')C(k; t, t'), \tag{3}$$

for isotropic, homogeneous turbulence.

Also, we may anticipate the introduction of a unimodal response tensor $R_{\alpha\alpha'}(\mathbf{k}; t, t')$, for $t > t'$, and state its isotropic form as

$$R_{\alpha\alpha'}(\mathbf{k}; t, t') = P_{\alpha\alpha'}(\mathbf{k})R(k; t, t'), \tag{4}$$

where $R(k; t, t')$ is called the response function.

In order to define the ensemble (and, when required, to study stationary turbulence) we introduce stirring forces. These are denoted by $f_\alpha(\mathbf{k}, t)$ and can be added to the right-hand side of the equation of motion, as we have done here in (2). These forces must be chosen to be isotropic, homogeneous and (in order to maintain incompressibility) solenoidal. It is usual to consider random forces with a multivariate normal probability distribution such that the associated functional integrals are analytically tractable. It is also usual to assume that the autocorrelation of the forces is instantaneous in time and we represent this by choosing the time autocorrelation to be a delta function. A form of correlation which satisfies all these requirements is

$$\langle f_\alpha(\mathbf{k}, t) f_\beta(\mathbf{k}', t')\rangle = P_{\alpha\beta}(\mathbf{k}) W(k)\delta(\mathbf{k} + \mathbf{k}')\delta(t - t'). \tag{5}$$

Here $W(k)$ is a spectral energy density which is related to the rate at which the force does work on the fluid.

3. Statistical formulation of the problem

We characterize the system by the exact probability distribution functional (pdf) which we denote by $P[\mathbf{u}, t]$, or P for short. It may be defined, in the language of statistical mechanics, as $P[\mathbf{u}, t]$ = the probability that the 'phase' of the system lies between $\mathbf{u}(\mathbf{k}, t)$ and $\mathbf{u}(\mathbf{k}, t) + d\mathbf{u}(\mathbf{k}, t)$. It is this pdf which we require in order to evaluate the covariance $C_{\alpha\beta}(\mathbf{k}, \mathbf{k}'; t, t')$, and other associated statistical quantities. Such averages will be represented by angle brackets.

The Liouville equation, which expresses conservation of probability, may be written [3, 18] as

$$\frac{\partial P[\mathbf{u}, t]}{\partial t} + \int \mathrm{d}^3 p \left\{ M_{\sigma\rho\delta}(\mathbf{p}) \int \mathrm{d}^3 n u_\rho(\mathbf{n}, t) u_\delta(\mathbf{m}, t) - \nu p^2 u_\sigma(\mathbf{p}, t) - \frac{\partial}{\partial u_\sigma(-\mathbf{p}, t)} W(k) \right\} \frac{\partial P[\mathbf{u}, t]}{\partial u_\sigma(\mathbf{p}, t)} = 0, \tag{6}$$

where $\mathbf{m} = \mathbf{p} - \mathbf{n}$. This equation was solved using perturbation methods by Edwards [3], for the stationary case where $\partial P/\partial t = 0$, and by a variant of this method by Herring, for both the stationary case [4] and the nonstationary case [5]. For later use, it will be helpful to write it (with some rearrangement) in the symbolic form:

$$\partial P/\partial t + L_W P + LP + VP = 0, \tag{7}$$

where

$$L_W P = -\int \mathrm{d}^3 p \frac{\partial}{\partial u_\sigma(\mathbf{p}, t)} W(k) \frac{\partial P[\mathbf{u}, t]}{\partial u_\sigma(-\mathbf{p}, t)}; \qquad LP = -\int \mathrm{d}^3 p \nu p^2 u_\sigma(\mathbf{p}, t) \frac{\partial P[\mathbf{u}, t]}{\partial u_\sigma(\mathbf{p}, t)}, \tag{8}$$

and

$$VP = \int \mathrm{d}^3 p\, M_{\sigma\rho\delta}(\mathbf{p}) \int \mathrm{d}^3 n\, u_\rho(\mathbf{n}, t) u_\delta(\mathbf{m}, t) \frac{\partial P[\mathbf{u}, t]}{\partial u_\sigma(\mathbf{p}, t)}. \tag{9}$$

In the work presented here we shall consider the nonstationary case and hence we will put $W(k) = 0$ and so $L_W P = 0$. This means that we are considering freely decaying turbulence, but our approach will also be valid for the stationary case.

4. Perturbative solution of the Liouville equation

The technique pioneered by Edwards is rather subtle, and relies on underlying symmetry considerations, which only emerge when one works through the analysis in some detail. Accordingly, it may be helpful to make the symmetry aspects manifest from the beginning, and this we now do. Let us anticipate that the probability distribution functional may be found as an approximation, in the form of the expansion:

$$P[\mathbf{u}, t] = P_0[\mathbf{u}] + \epsilon P_1[\mathbf{u}] + \epsilon^2 P_2[\mathbf{u}] + \cdots, \tag{10}$$

where $\epsilon = 1$ is a book-keeping parameter. Note that the coefficients P_1, P_2, P_3, and so on, are invariant distributions, the unsteady nature of the full distribution P being represented by changes in their relative magnitudes as time goes on. We now decompose the pdf into two parts which are respectively symmetric and antisymmetric, under the interchange of $\mathbf{u}$ and $-\mathbf{u}$; thus,

$$P = P_S + P_A. \tag{11}$$

These can be interpreted in terms of expansion (10), where P_S consists of the even-order terms $P_0, P_2, \ldots$ while P_A comprises the odd-order terms. Note that the symmetric part of the distribution P_S determines the even-order moments, whereas the antisymmetric part P_A determines the odd-order moments.

Now we introduce a model for the pdf. We follow the example of Edwards [3] and choose a model $\overline{P}_S$ which will give the exact result for the velocity field covariance. In addition, we shall require it to be mathematically tractable, such that we can perform functional integrals against even-order products of the velocity field. That is, $\overline{P}_S$ is chosen to be Gaussian and to possess the following properties:

$$\int \mathcal{D}\mathbf{u}\overline{P}_S[\mathbf{u}] = 1; \qquad \int \mathcal{D}\mathbf{u}\overline{P}_S[\mathbf{u}]\mathbf{u}\mathbf{u} = \langle \mathbf{u}\mathbf{u} \rangle. \tag{12}$$

Correspondingly, instead of the exact P_A, we work with its approximation $\overline{P}_A$, which is calculated from the Liouville equation using $\overline{P}_S$ to approximate the exact P_S. Hence we may write an approximate expression for the triple moment as

$$\int \mathcal{D}\mathbf{u}\overline{P}_A[\mathbf{u}]\mathbf{u}\mathbf{u}\mathbf{u} = \langle \mathbf{u}\mathbf{u}\mathbf{u} \rangle. \tag{13}$$

It should be noted that the assumption of Gaussian form implies that $\overline{P}_S$ does not possess the correct flatness factor and to obtain this we require a correction $\Delta\overline{P}_S$, say. In effect, we can identify $\overline{P}_S$, $\overline{P}_A$ and $\Delta\overline{P}_S$, with P_0, P_1 and P_2, and so we replace equation (11) by

$$P[\mathbf{u}, t] = \overline{P}_S[\mathbf{u}] + \epsilon\overline{P}_A[\mathbf{u}] + \epsilon^2\Delta\overline{P}_S[\mathbf{u}], \tag{14}$$

We may complete the specification of our zero-order model, by introducing an operator $\overline{L}_S$, such that it generates our model pdf $\overline{P}_S$; thus,

$$\frac{\partial\overline{P}_S}{\partial t} + \overline{L}_S\overline{P}_S = 0. \tag{15}$$

Now, we rewrite the Liouville equation (7) with the work term set equal to zero; in order to adapt this to a form suitable for perturbation theory, we add and subtract our model operator $\overline{L}_S$, leaving the equation unchanged. Then, we assign the order ϵ to the term VP as it is antisymmetric in the velocity fields, and order ϵ^2 to the difference term $[L - \overline{L}_S]P$, as this represents the correction to the flatness factor as given by $\overline{P}_S$. Thus we may write (7) as

$$\frac{\partial P}{\partial t} + \overline{L}_S P + \epsilon V P + \epsilon^2[L - \overline{L}_S]P = 0, \tag{16}$$

substitute from (14) for P, and collect terms at each order of ϵ to obtain
ϵ^0:

$$\frac{\partial \overline{P}_S}{\partial t} + \overline{L}_S \overline{P}_S = 0; \tag{17}$$

ϵ^1:

$$\frac{\partial \overline{P}_A}{\partial t} + V \overline{P}_S = 0; \tag{18}$$

and so on.

The key equation here is (18) at order ϵ which we can write out in full and integrate with respect to time as

$$\overline{P}_A = -\int \mathrm{d}s \int \mathrm{d}^3 p \int \mathrm{d}^3 n \, M_{\sigma\rho\delta}(\mathbf{p}) u_\rho(\mathbf{n}, s) u_\delta(\mathbf{m}, s) \times \frac{\partial \overline{P}_S}{\partial u_\sigma(\mathbf{p}, s)}. \tag{19}$$

Note that we may check the consistency of our procedures by noting that this form for $\overline{P}_A$ is antisymmetric under the interchange of $\mathbf{u}$ with $-\mathbf{u}$. Similarly, it may be verified that $\Delta \overline{P}_S$ is symmetric, consisting, as it does, of terms like $V^2 \overline{P}_S$.

5. The response function as a mean-field approximation

We may now use our approximation to the full pdf to obtain an equation for the two-time covariance in the usual way from the Navier–Stokes equation, as given by (2); thus,

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \nu k^2\right) C\left(k; t, t'\right) &= \frac{1}{2} M_{\alpha\beta\gamma}(\mathbf{k}) \int \mathrm{d}^3 j \int \mathcal{D}\mathbf{u} \overline{P}_A \, u_\beta(\mathbf{j}, t) u_\gamma(\mathbf{l}, t) u_\alpha(-\mathbf{k}, t') \\ &= \frac{1}{2} M_{\alpha\beta\gamma}(\mathbf{k}) \int \mathrm{d}^3 j \int \mathcal{D}\mathbf{u} \int \mathrm{d}s \int \mathrm{d}^3 p \, M_{\sigma\rho\delta}(\mathbf{p}) \int \mathrm{d}^3 n \\ &\quad \times u_\rho(\mathbf{n}, s) u_\delta(\mathbf{m}, s) \overline{P}_S \frac{\partial}{\partial u_\sigma(\mathbf{p}, s)} \times [u_\beta(\mathbf{j}, t) u_\gamma(\mathbf{l}, t) u_\alpha(-\mathbf{k}, t')].\end{aligned} \tag{20}$$

Note that in going from the first line to the second, we performed a partial functional integration and dropped the resulting fifth-order moment, as this is zero when evaluated against an even distribution. Also note that when the functional differential lands on the two velocity fields to its left, the differentiation is at the same time and hence generates the usual delta functions (e.g. $\delta(\mathbf{n} - \mathbf{p})$). As these are incompatible with the triangle relationships the resulting correlations give zero and so these two terms do not contribute.

Now let $T(k; t, t')$ stand for the right-hand side of (20), and denote the ground-state average against $\overline{P}_S$ by $\langle \cdots \rangle_S$. Then we may write

$$\begin{aligned}T(k; t, t') = \frac{1}{2} M_{\alpha\beta\gamma}(\mathbf{k}) \int \mathrm{d}^3 j \int \mathrm{d}s \int \mathrm{d}^3 p \, M_{\sigma\rho\delta}(\mathbf{p}) \int \mathrm{d}^3 n \\ \times \left\langle u_\rho(\mathbf{n}, s) u_\delta(\mathbf{m}, s) \frac{\partial}{\partial u_\sigma(\mathbf{p}, s)} [u_\beta(\mathbf{j}, t) u_\gamma(\mathbf{l}, t) u_\alpha(-\mathbf{k}, t')] \right\rangle_S.\end{aligned} \tag{21}$$

Differentiating each member of the triple product in turn then gives

$$\begin{aligned}T(k; t, t') = \frac{1}{2} M_{\alpha\beta\gamma}(\mathbf{k}) \int \mathrm{d}^3 j \int \mathrm{d}s \int \mathrm{d}^3 p \, M_{\sigma\rho\delta}(\mathbf{p}) \int \mathrm{d}^3 n \\ \times \left[\left\langle u_\rho(\mathbf{n}, s) u_\delta(\mathbf{m}, s) u_\gamma(\mathbf{l}, t) u_\alpha(-\mathbf{k}, t') \frac{\partial u_\beta(\mathbf{j}, t)}{\partial u_\sigma(\mathbf{p}, s)} \right\rangle_S \right.\end{aligned}$$

$$+\left\langle u_\rho(\mathbf{n},s)u_\delta(\mathbf{m},s)u_\beta(\mathbf{j},t)u_\alpha(-\mathbf{k},t')\frac{\partial u_\sigma(\mathbf{l},t)}{\partial u_\sigma(\mathbf{p},s)}\right\rangle_S$$

$$+\left\langle u_\rho(\mathbf{n},s)u_\delta(\mathbf{m},s)u_\beta(\mathbf{j},t)u_\gamma(\mathbf{l},t)\frac{\partial u_\alpha(-\mathbf{k},t')}{\partial u_\sigma(\mathbf{p},s)}\right\rangle_S\Bigg]. \tag{22}$$

At this point we make the assumption that the functional derivative can be treated as statistically independent of the velocity field. This is, in effect, a mean-field approximation and is very similar to the corresponding step taken by Kraichnan in the derivation of the direct interaction approximation (DIA) [1]: also see page 56 in [18] or page 375 in [19]. Then, averages can be evaluated in terms of the covariances using the properties of $\overline{P}_S$. Homogeneity leads to delta functions, which are integrated out, along with the integrals with respect to $\mathbf{p}$ and $\mathbf{n}$.

As an example, we will work out the average in the first term on the right-hand side of equation (22), as follows:

$$\int d^3p\, M_{\sigma\rho\delta}(\mathbf{p})\int d^3n\left\langle u_\rho(\mathbf{n},s)u_\delta(\mathbf{m},s)u_\gamma(\mathbf{l},t)u_\alpha(-\mathbf{k},t')\frac{\partial u_\beta(\mathbf{j},t)}{\partial u_\sigma(\mathbf{p},s)}\right\rangle_S$$

$$= 2\int d^3p M_{\sigma\rho\delta}(\mathbf{p})\int d^3n\langle u_\rho(\mathbf{n},s)u_\gamma(\mathbf{l},t)\rangle_S\langle u_\delta(\mathbf{m},s)u_\alpha(-\mathbf{k},t')\rangle_S\left\langle\frac{\partial u_\beta(\mathbf{j},t)}{\partial u_\sigma(\mathbf{p},s)}\right\rangle_S$$

$$= 2\int d^3p M_{\sigma\rho\delta}(\mathbf{p})\int d^3n C_{\rho\gamma}(\mathbf{l};t,s)C_{\delta\alpha}(-\mathbf{k};t',s)\delta(\mathbf{n}+\mathbf{l})\delta(\mathbf{m}-\mathbf{k})\left\langle\frac{\partial u_\beta(\mathbf{j},t)}{\partial u_\sigma(\mathbf{p},s)}\right\rangle_S$$

$$= 2M_{\sigma\rho\delta}(\mathbf{j})C_{\rho\gamma}(\mathbf{k}-\mathbf{j};t,s)C_{\delta\alpha}(-\mathbf{k};t',s)\left\langle\frac{\partial u_\beta(\mathbf{j},t)}{\partial u_\sigma(\mathbf{j},s)}\right\rangle_S, \tag{23}$$

where the factor 2 arises because there is another possible pairing. Note that some pairings are forbidden: for instance, $\mathbf{j}=-\mathbf{l}$ violates the triangle condition, and hence does not contribute. Also, recall that $\mathbf{m}=\mathbf{p}-\mathbf{n}$ and $\mathbf{l}=\mathbf{k}-\mathbf{j}$, so that the integration with respect to $\mathbf{n}$ yields the delta function $\delta(\mathbf{p}-\mathbf{j})$, and hence the integral with respect to $\mathbf{p}$ picks out the components of the functional derivative on the wavenumber diagonal.

Next we define the response tensor $R_{\alpha\alpha'}(\mathbf{k};t,t')$ in terms of the functional derivative; thus,

$$\left\langle\frac{\partial u_\alpha(\mathbf{k},t)}{\partial u_{\alpha'}(\mathbf{k},t')}\right\rangle_S = R_{\alpha\alpha'}(\mathbf{k};t,t') \qquad \text{for} \quad t>t', \tag{24}$$

and substitute accordingly in each of the three terms on the right-hand side of equation (22). Each of the three terms is multiplied by a factor of 2, as we have just seen for the first one, and we can also rename dummy variables so that we can replace the sum of the first and second terms by twice the first. Invoking the isotropic forms for the covariance and response tensors, collecting terms and re-naming dummy variables as appropriate, we substitute back into (20) to obtain the two-time covariance equation in the form

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)C(k;t,t') = \int d^3j\, L(\mathbf{k},\mathbf{j})\left[\int_0^{t'} ds R(k;t',s)C(j;t,s)C(|\mathbf{k}-\mathbf{j}|;t,s)\right.$$
$$\left. - \int_0^t ds R(j;t,s)C(k;s,t')C(|\mathbf{k}-\mathbf{j}|;t,s)\right], \tag{25}$$

where $L(\mathbf{k},\mathbf{j})$ is defined by $L(\mathbf{k},\mathbf{j}) = -2M_{\alpha\beta\gamma}(\mathbf{k})M_{\beta\alpha\delta}(\mathbf{j})P_{\gamma\delta}(\mathbf{k}-\mathbf{j})$. We may also derive an analogous equation for the single-time $C(k;t,t)$; thus,

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right)C(k;t,t) = 2\int d^3j L(\mathbf{k},\mathbf{j})\left[\int_0^t ds\, R(k;t,s)C(j;t,s)C(|\mathbf{k}-\mathbf{j}|;t,s)\right.$$
$$\left. - \int_0^t ds\, R(j;t,s)C(k;s,t)C(|\mathbf{k}-\mathbf{j}|;t,s)\right]. \tag{26}$$

Equations (25) for $C(k; t, t')$ and (26) for $C(k; t, t)$ are identical in form to the Kraichnan–Wyld covariance equations [1, 2], but with one crucial difference. Here the response of the system is not determined by the variation of the velocity field due to an infinitesimal change in the stirring force. Instead the response function arises quite naturally when we carry out the functional differentiation with respect to changes in the velocity field.

6. Conclusion

From equation (24) we have the definition of the response function. Noting that homogeneity has imposed a unimodal structure on this, and that isotropy ensures that its tensor character can be represented by the projector $P_{\alpha\beta}(\mathbf{k})$, our expression for the response function is

$$R(k; t, t') = -\int \mathcal{D}\mathbf{u}(\mathbf{k}, t)u_\alpha(\mathbf{k}, t) \times \frac{\partial \overline{P}_S[\mathbf{u}(\mathbf{k}, t), t]}{\partial u_\alpha(\mathbf{k}, t)}, \tag{27}$$

the last step following by partial integration. This result, although derived by very different methods, is the same as the general form given by [13, 14]. As $\overline{P}_S$ is a stationary Gaussian distribution, it follows that (27) reduces to the linear form given by (1). We note that although this is a linear result, when it is used in conjunction with equations (25) and (26) for the covariances, the resulting statistical closure is fully nonlinear. It should also be noted that in this sense the linear fluctuation–dissipation relation holds to all orders of renormalized perturbation theory.

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